

In[247]:= 2 - 2

Out[247]= 0

## Dove Prism: 720° per rotation

Nicholas Wheeler

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In[248]:=

### ■ Introduction

In class I have had occasion to discuss the double-valuedness of the  $SU(2)$  representation of  $O(3)$ . On such occasions I have usually demonstrated variants of the "Dirac string trick" (the "belt trick," the "plate trick," the "spinor spanner," an arrangement for supplying power to an appliance mounted on a rotating platform without having recourse to slip-rings, the design of a self-exciting dynamo without slip-rings, an invariable property of braided leather belts, etc.). And I have demonstrated the surprising action of Dove prisms, which cause the transmitted image to execute two revolutions when the prism itself is rotated once about its optical axis. [Heinrich Wilhelm Dove (1803-1879) is remembered primarily for his contributions to meteorology.]

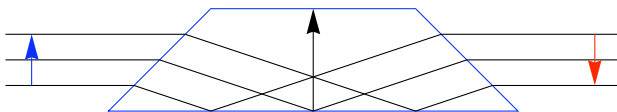
It occurred a few days ago to my wondering mind that I had never seen or produced an account of why the Dove prism does what it does. My primary objective here is to provide such an account.

In[249]:=

### ■ Ray transmission through a Dove prism

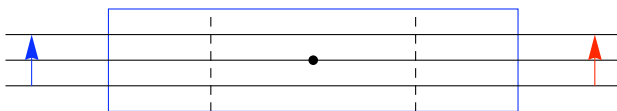
```
In[250]:= RaysInActivePlane = Graphics[{{Blue, Line[{{0, 0}, {4, 0}, {3, 1}, {1, 1}, {0, 0}]}},  
    Line[{{-1, 0.5}, {0.5, 0.5}, {2, 0}, {3.5, 0.5}, {5, 0.5}]}},  
    Line[{{-1, 0.75}, {0.75, 0.75}, {3, 0}, {3.75, 0.25}, {5, 0.25}]}},  
    Line[{{-1, 0.25}, {0.25, 0.25}, {1.0, 0}, {3.25, 0.75}, {5, 0.75}]}},  
    {Arrow[{{2, 0}, {2, 1}]}},  
    {Blue, Arrow[{{-0.75, 0.25}, {-0.75, 0.75}]}},  
    {Red, Arrow[{{4.75, 0.75}, {4.75, 0.25}]}}]}
```

Out[250]=



```
In[251]:= RaysInTransversePlane =  
    Graphics[  
        {Blue, Line[{{0, 0}, {4, 0}, {4, 1}, {0, 1}, {0, 0}]}},  
        {Dashed, Line[{{1, 0}, {1, 1}]}},  
        {Dashed, Line[{{3, 0}, {3, 1}]}},  
        Line[{{-1, 0.25}, {5, 0.25}]}},  
        Line[{{-1, 0.50}, {5, 0.50}]}},  
        Line[{{-1, 0.75}, {5, 0.75}]}},  
        {PointSize[0.015], Point[{2, 0.5}]}},  
        {Blue, Arrow[{{-0.75, 0.25}, {-0.75, 0.75}]}},  
        {Red, Arrow[{{4.75, 0.25}, {4.75, 0.75}]}}]}
```

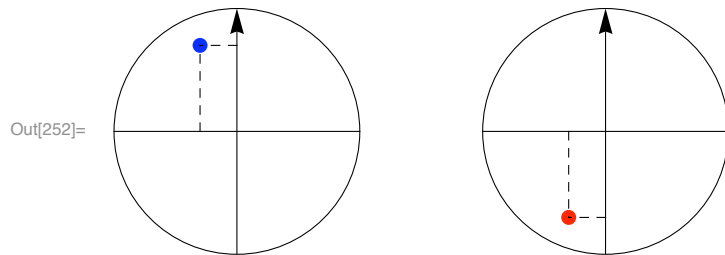
Out[251]=



Let the **Blue Arrow** represent the image presented to the prism, and the **Red Arrow** the image perceived by the downstream viewer.

In the following figure the black arrow indicates the orientation of the prism. The blue dot ● refers to a point on the imaged presented to the prism, the red dot ● refers to its transmitted location:

```
In[252]:= Graphics[{
  Circle[{0, 0}, 1],
  Circle[{3, 0}, 1],
  Arrow[{{0, -1}, {0, 1}}],
  Line[{{-1, 0}, {1, 0}}],
  Arrow[{{3, -1}, {3, 1}}],
  Line[{{2, 0}, {4, 0}}],
  {Blue, PointSize[Large], Point[{-0.3, 0.7}]},
  {Dashed, Line[{{0, 0.7}, {-0.3, 0.7}]}},
  {Dashed, Line[{{-0.3, 0}, {-0.3, 0.7}]}},
  {Red, PointSize[Large], Point[{2.7, -0.7}]},
  {Dashed, Line[{{3, -0.7}, {2.7, -0.7}]}},
  {Dashed, Line[{{2.7, 0}, {2.7, -0.7}]}]}
}]
```



■ The effect of axial rotation of the prism

Let the unit vector

$$\mathbf{z}[\alpha] := \begin{pmatrix} \cos\left[\frac{\pi}{2} + \alpha\right] \\ \sin\left[\frac{\pi}{2} + \alpha\right] \end{pmatrix}$$

describe the orientation of the prism. At  $\alpha = 0$  the  $\mathbf{z}$ -vector coincides with the  $z$ -axis, and  $\alpha$  measures counterclockwise rotation away from the  $z$ -axis.

```
In[254]:= z[0] // MatrixForm
```

$$\text{Out[254]//MatrixForm} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The matrix

```
In[255]:= Z[alpha] := z[alpha].Transpose[z[alpha]]
```

```
In[256]:= Z[α] // Simplify // MatrixForm
Eigenvalues[Z[α]]
```

```
Out[256]/MatrixForm=
```

$$\begin{pmatrix} \sin[\alpha]^2 & -\cos[\alpha] \sin[\alpha] \\ -\cos[\alpha] \sin[\alpha] & \cos[\alpha]^2 \end{pmatrix}$$

```
Out[257]= {1, 0}
```

```
In[258]:= Z[α].z[α] == z[α] // Simplify
```

```
Out[258]= True
```

projects onto  $\mathbf{z}[\alpha]$  and

```
In[259]:= I =  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ;
```

```
N[α_] := I - Z[α]
```

```
N[α] // Simplify // MatrixForm
```

```
Eigenvalues[N[α]]
```

```
N[α].z[α] // Simplify // MatrixForm
```

```
Out[261]/MatrixForm=
```

$$\begin{pmatrix} \cos[\alpha]^2 & \cos[\alpha] \sin[\alpha] \\ \cos[\alpha] \sin[\alpha] & \sin[\alpha]^2 \end{pmatrix}$$

```
Out[262]= {1, 0}
```

```
Out[263]/MatrixForm=
```

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

projects onto the ray normal to  $\mathbf{z}[\alpha]$ .

Let

```
In[264]:= s =  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ;
```

be a source vector presented to the prism. The perceived output will then be

```
In[265]:= P[α_] := N[α].s - Z[α].s
```

which in terms of

```
In[266]:= Q[α_] := N[α] - Z[α]
```

```
Simplify[Q[α]] // Simplify // MatrixForm
```

```
Out[267]/MatrixForm=
```

$$\begin{pmatrix} \cos[2\alpha] & \sin[2\alpha] \\ \sin[2\alpha] & -\cos[2\alpha] \end{pmatrix}$$

**NOTE:** In the previously distributed version of this notebook I used an old version of that command, which produced a matrix missing the 2s, which are the point of the story.

—which is a **double-angle rotation matrix**

```
In[268]:= Q[α].Transpose[Q[α]] // Simplify // MatrixForm
```

```
Out[268]/MatrixForm=
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

becomes

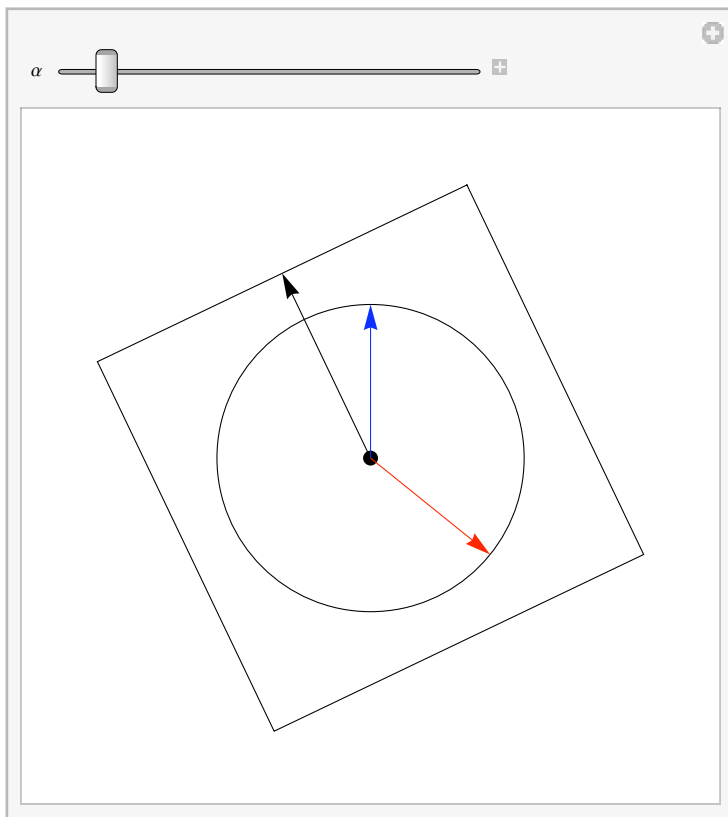
```
In[269]:= Q[α].s // Simplify // MatrixForm
```

```
Out[269]/MatrixForm=
```

$$\begin{pmatrix} \sin[2\alpha] \\ -\cos[2\alpha] \end{pmatrix}$$

```
In[270]:= Manipulate[Graphics[{
  Line[{{{\sqrt{2} Cos[\frac{\pi}{4} + \alpha], \sqrt{2} Sin[\pi/4 + \alpha]},
    {\sqrt{2} Cos[\frac{\pi}{4} + \frac{\pi}{2} + \alpha], \sqrt{2} Sin[\frac{\pi}{4} + \frac{\pi}{2} + \alpha]},
    {\sqrt{2} Cos[\frac{\pi}{4} + 2 \frac{\pi}{2} + \alpha], \sqrt{2} Sin[\frac{\pi}{4} + 2 \frac{\pi}{2} + \alpha]},
    {\sqrt{2} Cos[\frac{\pi}{4} + 3 \frac{\pi}{2} + \alpha], \sqrt{2} Sin[\frac{\pi}{4} + 3 \frac{\pi}{2} + \alpha]},
    {\sqrt{2} Cos[\frac{\pi}{4} + 4 \frac{\pi}{2} + \alpha], \sqrt{2} Sin[\frac{\pi}{4} + 4 \frac{\pi}{2} + \alpha]}}}],
  {White, Circle[{0, 0}, 1.5]},
  Arrow[{{0, 0}, Flatten[z[α]]}],
  PointSize[Large], Point[{0, 0}],
  {Blue, Arrow[{{0, 0}, 0.75 Flatten[s]}]},
  {Red, Arrow[{{0, 0}, 0.75 Flatten[Q[α].s]}]},
  Circle[{0, 0}, 0.75]}, {α, 0, 2 π}]
```

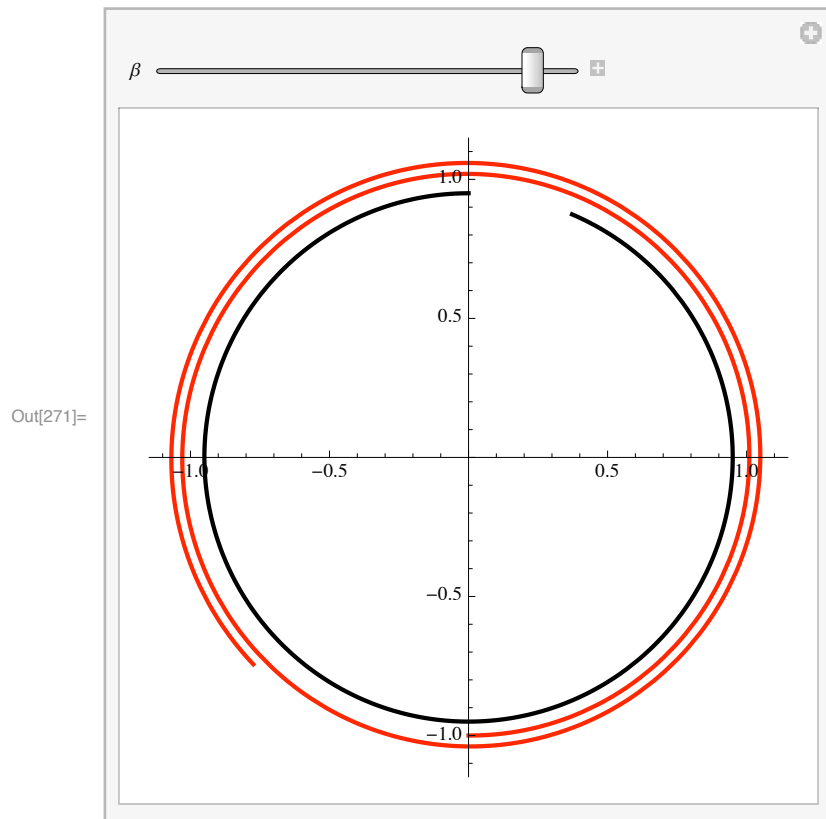
```
Out[270]=
```



The graphic demonstrates that a  $360^\circ$  rotation of the prism about its optical axis causes the red image of the fixed blue source executes a  $720^\circ$  excursion. Below is another demonstration of that same fact:

**REMARK:** Here I have animated what in both previous versions of this notebook was a static figure. If you open the controls then each click on the + button advances the black circle by  $\pi/8 =$  one sixteenth of a complete cycle.

```
In[271]:= Manipulate[
  ParametricPlot[{{-0.95 Sin[α], 0.95 Cos[α]}, {-Sin[π + 2 α] (1 + α/80), Cos[π + 2 α] (1 + α/80)}},
    {α, 0, β}, PlotRange → {{-1.15, 1.15}, {-1.15, 1.15}},
    PlotStyle → {{Thick, Black}, {Thick, Red}}, {β, -.01, 2 π, π/8}]
```



In[272]:=

### ■ How does a Dove prism affect the polarization of a transmitted beam?

The question is a natural one, and the answer (submit keywords "dove prism polarization" to Google) appears to be not at all obvious or easy to obtain, surprising (transmission rotates the image, but not—or only relatively slightly—the polarization), and to have a number of significant applications.

### ■ Rotational transformation of a spinor

From the (obviously traceless) antisymmetric matrix

$$\text{In[273]:= } \mathbb{A} = \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

generate the rotation matrix

```
In[274]:= R[ $\alpha$ _] := MatrixExp[A]
R[ $\alpha$ ] // MatrixForm
```

```
Out[275]//MatrixForm=
  ( Cos[ $\alpha$ ]  -Sin[ $\alpha$ ] )
  ( Sin[ $\alpha$ ]   Cos[ $\alpha$ ] )
```

which acts upon the 2-vector

```
In[276]:= x = (  $x_1$  );
              (  $x_2$  );
```

to produce

```
In[277]:= R[ $\alpha$ ].x
```

```
Out[277]= {{Cos[ $\alpha$ ]  $x_1$  - Sin[ $\alpha$ ]  $x_2$ }, {Sin[ $\alpha$ ]  $x_1$  + Cos[ $\alpha$ ]  $x_2$ }}
```

```
In[278]:= y[ $\alpha$ _] := {{Cos[ $\alpha$ ]  $x_1$  - Sin[ $\alpha$ ]  $x_2$ }, {Sin[ $\alpha$ ]  $x_1$  + Cos[ $\alpha$ ]  $x_2$ }}
y[ $\alpha$ ] // MatrixForm
```

```
Out[279]//MatrixForm=
  ( Cos[ $\alpha$ ]  $x_1$  - Sin[ $\alpha$ ]  $x_2$  )
  ( Sin[ $\alpha$ ]  $x_1$  + Cos[ $\alpha$ ]  $x_2$  )
```

the elements of which we denote

```
In[280]:= y1[ $\alpha$ _] := Cos[ $\alpha$ ]  $x_1$  - Sin[ $\alpha$ ]  $x_2$ 
y2[ $\alpha$ _] := Sin[ $\alpha$ ]  $x_1$  + Cos[ $\alpha$ ]  $x_2$ 
```

```
In[282]:=
```

Now write

```
In[283]:= I = (  $1$    $0$  );
              (  $0$    $1$  );
```

from the traceless antihermitian matrix

```
In[284]:= i  $\alpha$  PauliMatrix[3]
```

```
Out[284]= {{ $i$   $\alpha$ , 0}, {0, - $i$   $\alpha$ }}
```

```
In[285]:= G[ $\alpha$ _] := {{ $i$   $\alpha$ , 0}, {0, - $i$   $\alpha$ }}
```

generate the unitary matrix

```
In[286]:= MatrixExp[G[ $\alpha$ ]]
```

```
Out[286]= {{ $e^{i \alpha}$ , 0}, {0,  $e^{-i \alpha}$ }}
```

```
In[287]:= U[ $\alpha$ _] := {{ $e^{i \alpha}$ , 0}, {0,  $e^{-i \alpha}$ }}
U[ $\alpha$ ] // MatrixForm
```

```
Out[288]//MatrixForm=
  (  $e^{i \alpha}$   0 )
  ( 0           $e^{-i \alpha}$  )
```

and note in passing that

```
In[289]:= U[0] == I
          U[π] == -I
          U[2 π] == I
          Inverse[U[α]] == U[-α]
```

```
Out[289]= True
```

```
Out[290]= True
```

```
Out[291]= True
```

```
Out[292]= True
```

From the elements of  $\mathbb{X}$  construct the Hermitian matrix (2nd-rank spinor)

```
In[293]:= X = x1 PauliMatrix[1] + x2 PauliMatrix[2];
          X // MatrixForm
```

```
Out[294]/MatrixForm=
  (  0      x1 - i x2 )
  ( x1 + i x2    0    )
```

and look finally to its transform

```
In[295]:= Inverse[U[α]].X.U[α] // ComplexExpand
```

```
Out[295]= {{0, Cos[2 α] x1 - Sin[2 α] x2 + i (-Sin[2 α] x1 - Cos[2 α] x2)},
           {Cos[2 α] x1 - Sin[2 α] x2 + i (Sin[2 α] x1 + Cos[2 α] x2), 0}}
```

```
In[296]:= Y[α_] := {{0, Cos[2 α] x1 - Sin[2 α] x2 + i (-Sin[2 α] x1 - Cos[2 α] x2)},
                  {Cos[2 α] x1 - Sin[2 α] x2 + i (Sin[2 α] x1 + Cos[2 α] x2), 0}}
```

which, we observe, can be described (compare the description of  $\mathbb{X}$ )

```
In[297]:= Y[α] == y1[2 α] PauliMatrix[1] + y2[2 α] PauliMatrix[2] // Simplify
```

```
Out[297]= True
```

So the coordinates of  $\mathbb{Y}[\alpha]$  rotate, but at twice the rate of the coordinates of  $\mathbb{y}[\alpha]$ , just as the transmitted image rotates at twice the rate of the Dove prism itself.

Here follow some relevant observations

```
In[298]:= Y[0] == X
          Y[0] == X
```

```
Out[298]= True
```

```
Out[299]= True
```

```
In[300]:= Y[π/2]
          Y[π/2] == -X
```

```
Out[300]= {{-x2}, {x1}}
```

```
Out[301]= True
```

```
In[302]:= Y[π] == -X
          Y[π] == X
```

```
Out[302]= True
```

```
Out[303]= True
```

```
In[304]:=  $\mathbb{Y}[3 \pi / 2] == -\mathbb{Y}[\pi / 2]$   

 $\mathbb{Y}[3 \pi / 2] == -\mathbb{X}$ 
```

```
Out[304]= True
```

```
Out[305]= True
```

```
In[306]:=  $\mathbb{Y}[2 \pi] == \mathbb{X}$   

 $\mathbb{Y}[2 \pi] == \mathbb{X}$ 
```

```
Out[306]= True
```

```
Out[307]= True
```

and

```
In[308]:=  $\mathbb{U}[0] == \mathbb{I}$   

 $\mathbb{U}[\pi] == -\mathbb{I}$   

 $\mathbb{U}[2 \pi] == \mathbb{I}$ 
```

```
Out[308]= True
```

```
Out[309]= True
```

```
Out[310]= True
```

Underlying the fact that a single rotation of  $\mathbb{y}[\alpha]$  produces a double rotation of  $\mathbb{Y}[\alpha]$  is the fact that

```
In[311]:=  $\mathbb{U}[\alpha + \pi] == -\mathbb{U}[\alpha]$  // Simplify
```

```
Out[311]= True
```

$\mathbb{U}[\alpha]$  enters quadratically into the transformation

```
In[312]:=  $\mathbb{X} \rightarrow \mathbb{Y}[\alpha] = \mathbb{U}[-\alpha] \cdot \mathbb{X} \cdot \mathbb{U}[\alpha]$ 
```

```
Out[312]=  $\left\{ \left\{ 0, e^{-2i\alpha} (x_1 - i x_2) \right\}, \left\{ e^{2i\alpha} (x_1 + i x_2), 0 \right\} \right\}$ 
```

so is invariant under  $\mathbb{U} \rightarrow -\mathbb{U}$ .

The preceding is a special cases of what is usually presented as a relation between  $O(3)$  and  $SU(2)$ : we have looked to transformations in the  $\{x,y\}$ -plane that result from rotations about the  $z$ -axis.

```
In[313]:=
```



## ■ The Dirac string trick

The trick is easier to demonstrate physically than to render in a *Mathematica* notebook (unless one is prepared to enter into the algebraic theory of brades and knots). The subject is treated in Louis H. Kauffman's *Knots & Physics* (4th edition 2012) and on the web at

<https://arxiv.org/pdf/1508.06028>

in his essay "Knot Logic: Logical Connection and Topological Connection," which provides references to other discussions of the topic and alludes to a wide variety of related subjects. He also had a hand in the production of an elegant short film "Air on the Diract Strings" (1993), which can be found at

<https://www.elv.uic.edu/hypercomplex/html/dirac.html>

or by simply consulting Google. The film includes the performance of a Phillipine wine dance that incorporates the "plate trick" version of the string trick.

The string trick has nothing (at least nothing obvious) to do with the Dirac strings of monopole theory.

**NOTE:** A couple of weeks after this notebook was completed, I—on recommendation of R. Thomas Harris (Reed classmate, now a retired mathematician)—read Edward Frenkel's *Love & Math: The Heart of Hidden Reality*, an autobiographical account of his involvement with the "Langlands Program." On his pages 170-171 Frenkel provides an illustrated account of what he calls "the cup trick," which on page 276 (Notes 4 & 6) he reports that the Phillipine dance is called "Binasuan," and cites [http://www.youtube/watch?v=N2TOOz\\_eaTY](http://www.youtube/watch?v=N2TOOz_eaTY). He does so because the dance illustrates a simple instance of "Langlands duality."

## ■ String trick addendum

I have prepared a set of figures that indicate how the trick is performed. To the corners of triangular card (red on one side, blue on the other), attach red, blue and black strings as indicated in Figure 1. Consider the remote ends of the strings to be pinned.

A 360° clockwise rotation of the card produces a 360° left-handed twist in the red/blue strings. Experiment establishes that so long as the card remains red-face-up and the remote ends of the strings remain pinned there is no way to return the strings to their original no-string-crossings orientation.

A sequence of ten figures shows how after a 720° rotation the strings—with those constraints still in effect—can be restored to their original orientation.

```
In[314]:= Dirac1 = Graphics[{
  {Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
  {Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
```

```

{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},

{Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
{Thick, Blue, Line[{{-1, 4}, {-1, 0}}]},
{Thick, Red, Line[{{1, 4}, {1, 0}}]}
]];
Dirac2 = Graphics[{
  {Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
  {Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},
  {Blue, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},
  {Thick, Red, Line[{{1, 4}, {-1, 0}}]},
  {White, Disk[0, 2], 0.15}},
{Thick, Blue, Line[{{-1, 4}, {1, 0}}]}
]];
Dirac3 = Graphics[{
  {Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
  {Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},
  {Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},
  {Thick, Red, Line[{{1, 4}, {-1, 2}}]},
  {White, Disk[0, 3], 0.15}},
{Thick, Blue, Line[{{-1, 4}, {1, 2}}]},
{Thick, Blue, Line[{{1, 2}, {-1, 0}}]},
{White, Disk[0, 1], 0.15}},
{Thick, Red, Line[{{-1, 2}, {1, 0}}]}
]];
Dirac4 = Graphics[{
  {Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
  {Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},
  {Blue, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},
  {Thick, Red, Line[{{1, 4}, {-1, 8/3}}]},
  {White, Disk[0, 20/6], 0.15}},
{Thick, Blue, Line[{{-1, 4}, {1, 8/3}}]},
{Thick, Blue, Line[{{1, 8/3}, {-1, 4/3}}]},
{White, Disk[0, 2], 0.15}},
{Thick, Red, Line[{{-1, 8/3}, {1, 4/3}}]},
{Thick, Red, Line[{{1, 4/3}, {-1, 0}}]},
{White, Disk[0, 4/6], 0.15}},
{Thick, Blue, Line[{{-1, 4/3}, {1, 0}}]}
]];
Dirac5 = Graphics[{
  {Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
  {Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},
  {Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
  {Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},

```

```

{Thick, Red, Line[{{1, 4}, {-1, 3}}]},
{White, Disk[{0, 3.5}, 0.15]},
{Thick, Blue, Line[{{-1, 4}, {1, 3}}]},
{Thick, Blue, Line[{{1, 3}, {-1, 2}}]},
{White, Disk[{0, 2.5}, 0.15]},
{Thick, Red, Line[{{-1, 3}, {1, 2}}]},
{Thick, Red, Line[{{1, 2}, {-1, 1}}]},
{White, Disk[{0, 1.5}, 0.15]},
{Thick, Blue, Line[{{-1, 2}, {1, 1}}]},
{Thick, Blue, Line[{{1, 1}, {-1, 0}}]},
{White, Disk[{0, 0.5}, 0.15]},
{Thick, Red, Line[{{-1, 1}, {1, 0}}]}
]];

Dirac6 = Graphics[
{Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
{Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},
{Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},
{White, Disk[{0, -3.5}, 0.1]},
{Thick, Red, Line[{{1, 4}, {-1.5, 3}, {-1.5, -3.5}, {1.5, -3.5}, {1.5, 2}}]},
{White, Disk[{-0.1, 3.5}, 0.15]},
{Thick, Blue, Line[{{-1, 4}, {1, 3}}]},
{Thick, Blue, Line[{{1, 3}, {-1, 2}}]},
{Thick, Red, Line[{{-1, 1}, {1.5, 2}}]},
{White, Disk[{0.1, 1.5}, 0.15]},
{Thick, Blue, Line[{{-1, 2}, {1, 1}}]},
{Thick, Blue, Line[{{1, 1}, {-1, 0}}]},
{White, Disk[{0, 0.5}, 0.15]},
{Thick, Red, Line[{{-1, 1}, {1, 0}}]}
]];

Dirac7 = Graphics[
{Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
{Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},
{Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},
{White, Disk[{0, -3.5}, 0.1]},
{White, Disk[{0, -3}, 0.1]},
{Thick, Red, Line[{{1, 4}, {-1.5, 3}, {-1.5, -3.5}, {1.5, -3.5}, {1.5, 2}}]},
{White, Disk[{-0.1, 3.5}, 0.15]},
{Thick, Blue, Line[{{-1, 4}, {1, 3}, {-1.25, 2}, {-1.25, -3}, {1.25, -3}, {1.25, 1}}]},
{Thick, Red, Line[{{-1, 1}, {1.5, 2}}]},
{Thick, Blue, Line[{{1.25, 1}, {-1, 0}}]},
{White, Disk[{0.05, 0.5}, 0.15]},
{Thick, Red, Line[{{-1, 1}, {1, 0}}]}
]];

Dirac8 = Graphics[
{Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
{Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},

```

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{Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},
{White, Disk[{0, -3.5}, 0.1]},
{White, Disk[{0, -3}, 0.1]},
{Thick, Red, Line[{{1, 4}, {-1.5, 3}, {-1.5, -3.5}, {1.5, -3.5}, {1.5, 2}}]},
{White, Disk[{-0.1, 3.5}, 0.15]},
{Thick, Blue, Line[{{-1, 4}, {1, 3}, {-1.25, 2}, {-1.25, -3}, {1, -3}}]},
{Thick, Red, Line[{{-1, 1}, {1.5, 2}}]},
{White, Disk[{0.05, 0.5}, 0.15]},
{Thick, Red, Line[{{-1, 1}, {1, 0}}]},
{Thick, Blue, Line[{{1, -3}, {1, -2.5}, {-1, -2.5}, {-1, 0}}]},
{White, Disk[{0, -2.5}, 0.15]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -2.7}}]}
}];
Dirac9 = Graphics[
{Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
{Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
{Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
{Thick, Red, Line[{{1, 4}, {-1.5, 3}, {-1.5, -3.5}, {1.5, -3.5}, {1.5, 1.9}}]},
{White, Disk[{-0.1, 3.5}, 0.15]},
{Thick, Blue, Line[{{-1, 4}, {1, 3}, {-1.25, 2}, {-1.25, -3}, {1, -3}}]},
{Thick, Red, Line[{{-1, 1}, {1.5, 1.9}}]},
{White, Disk[{0.05, 0.5}, 0.15]},
{Thick, Red, Line[{{-1, 1}, {1, 0}}]},
{Thick, Blue, Line[{{1, -3}, {1, -2.5}, {-1, -2.5}, {-1, 0}}]},
{White, Disk[{0, -2.5}, 0.15]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -2.7},
{-1.125, -2.7}, {-1.125, 1.9}, {1.75, 2}, {1.75, -4}, {0, -4}, {0, -5}}]}
}];
Dirac10 = Graphics[
{Polygon[{{-1.05, 4}, {1.05, 4}, {1.05, 4.3}, {-1.05, 4.3}}]},
{Polygon[{{-1.05, -5}, {1.05, -5}, {1.05, -5.3}, {-1.05, -5.3}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -5}}]},
{Red, Polygon[{{-1.05, 0}, {0, -1.05 Sqrt[3]}, {1.05, 0}}]},
{Thick, Black, Line[{{0, -Sqrt[3]}, {0, -4}}]},
{Thick, Blue, Line[{{-1, 4}, {-1, 0}}]},
{Thick, Red, Line[{{1, 4}, {1, 0}}]}
}];

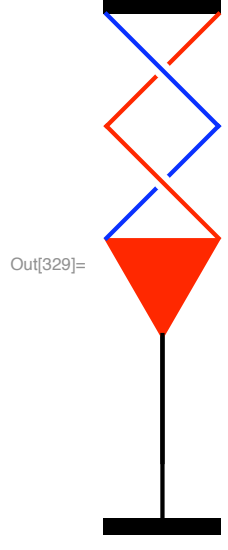
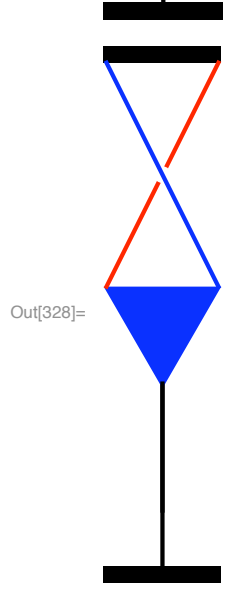
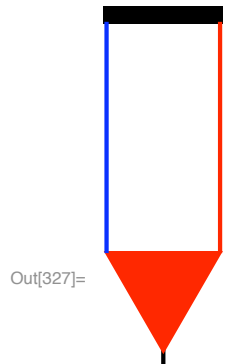
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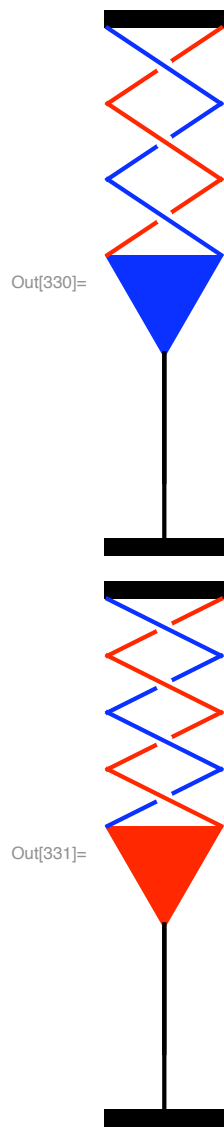
Page Break

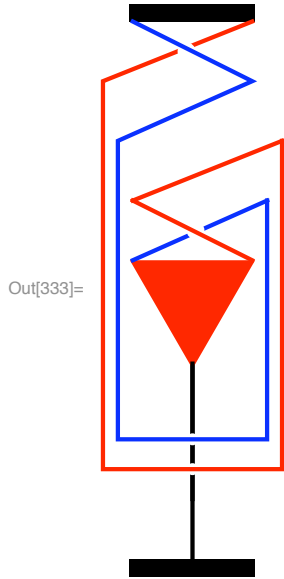
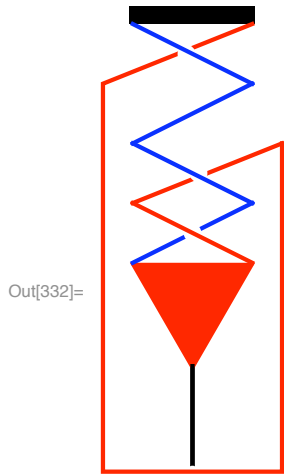
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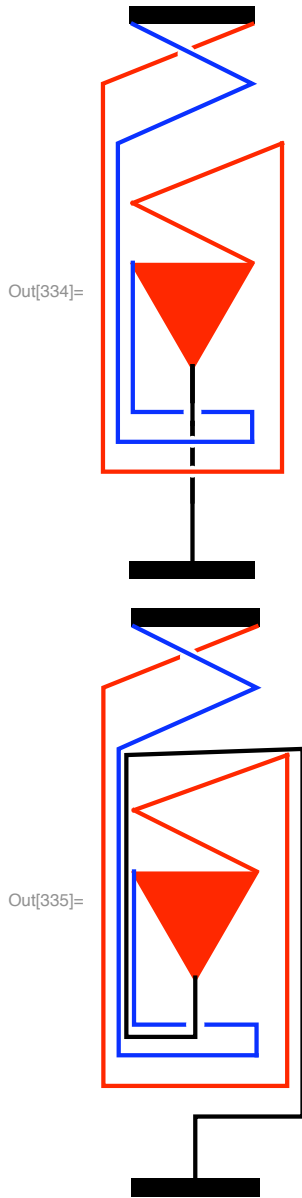
In[327]:= Show[Dirac1]
Show[Dirac2]
Show[Dirac3]
Show[Dirac4]
Show[Dirac5]
Show[Dirac6]
Show[Dirac7]
Show[Dirac8]
Show[Dirac9]
Show[Dirac10]

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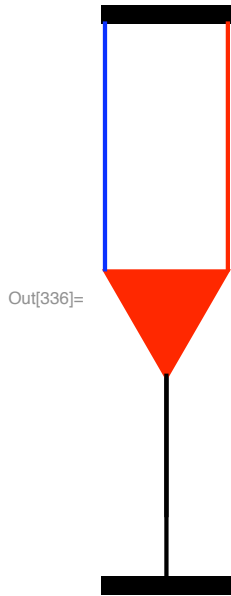












One can only wish that *Mathematica* provided a **Morph** command that permitted one to show in continuous time how each figure gives rise to the next.